International Journal of Theoretical Physics, Vol. 38, No. 5, 1999

($N + 1$)-Dimensional Quantum Mechanical Model for a Closed Universe

T. R. Mongan[1]

Received October 13, 1998

A quantum mechanical model for an ($N + 1$)-dimensional universe arising from a quantum fluctuation is outlined. (3 + 1) dimensions are a closed, infinitely expanding universe, and the remaining $N - 3$ dimensions are compact. The (3 + 1) noncompact dimensions are modeled by quantizing a canonical Hamiltonian description of a homogeneous isotropic universe. It is assumed that gravity and the strong-electroweak (SEW) force had equal strengths in the initial state. Inflation occurred when the compact ($N - 3$)-dimensional space collapsed after a quantum transition from the initial state of the universe during its evolution to the present state where gravity is much weaker than the SEW force. The model suggests the universe has no singularities and the large size of our present universe is determined by the relative strength of gravity and the SEW force today. A small cosmological constant, resulting from the zero-point energy of the scalar field corresponding to the compact dimensions, makes the model universe expand forever.

1. INTRODUCTION

This paper sketches a quantum mechanical model of an ($N + 1$)-dimensional universe (with $N > 6$) arising from a quantum fluctuation. In this model, (3 + 1) dimensions are a closed universe that is approximately a Friedmann–Robertson–Walker (FRW) universe in its early stages. The remaining $n = N - 3$ dimensions (called the C system) are compact, with a characteristic size equal to the Planck length.

The (3 + 1)-dimensional sector is treated differently from the usual quantum cosmology approach involving superspace (or minisuperspace) and the Wheeler–DeWitt equation.[1,2] Instead, the model uses a Schrödinger equation for the (3 + 1)-dimensional space like that obtained from the canonical Hamiltonian formulation of cosmology by Elbaz *et al.*[1] and

[1] Consulting Engineer, 84 Marin Avenue, Sausalito, California 94965; e-mail: jtm@crl.com.

0020-7748/99/0500-152 1$16.00/0

Novello *et al.*[2] In particular, refs. 1 and 2 demonstrate that the quantum dynamics of a homogeneous isotropic universe are equivalent to the quantum dynamics of a particle in a potential well.

This simple phenomenological model may provide some insight into (1) general features of more fundamental $(N+1)$-dimensional models of the forces governing the universe that allow the universe to arise from a quantum fluctuation and inflate to a radiation-dominated universe similar to the early stages of our universe (if our universe is closed), (2) the size of the universe (e.g., ref. 3), (3) how inflation starts and stops, and (4) the magnitude of the cosmological "constant."

The radius of curvature R of a closed FRW universe[4] obeys the Friedmann equation

$$\left(\frac{dR}{dt}\right)^2 - \left(\frac{8\pi G}{3}\right)\varepsilon\left(\frac{R}{c}\right)^2 = -c^2 \tag{1}$$

where ε is the mass-energy density of the universe, the gravitational constant $G = 6.67 \times 10^{-8}$ cm^3/g sec^2, and $c = 3 \times 10^{10}$ cm/sec. At present, the mass-energy density is $\varepsilon = \varepsilon_r(R_0/R)^4 + \varepsilon_m(R_0/R)^3$, where ε_r, ε_m, and R_0 are respectively the radiation and matter-energy density and the radius of curvature of the universe today. Multiplied by $\frac{1}{2}m$, equation (1) describes the motion of a particle with mass m and energy $\frac{1}{2}mc^2$ in the potential

$$V_R = -\frac{m}{2}\left(\frac{8\pi G}{3}\right)\left[\varepsilon_r\left(\frac{R_0}{R}\right)^4 + \varepsilon_m\left(\frac{R_0}{R}\right)^3\right]\left(\frac{R}{c}\right)^2$$

The constants in this potential can be approximated by taking $\varepsilon_r \approx 10^{-34}$ g/cm$^3 \cdot c^2$, $\varepsilon_m \approx 10^{-29}$ g/cm$^3 \cdot$ c^2, and $R_0 \approx 10^{28}$ cm, giving

$$V_R = -5.6\,\frac{m}{2}\left(\frac{10^{71}}{R^2} + \frac{10^{48}}{R}\right) \quad \text{g(cm/sec)}^2$$

The size of the constants in this potential, and the corresponding large size of our universe, has been a puzzle for years.[3] In the early universe, when $R << 10^{-5}R_0$, the radiation term in V_R dominated, so this paper focuses on the radiation-dominated universe.

The size of the compact dimensions corresponds to a gauge singlet scalar field ϕ in the FRW sector of the universe (e.g., ref. 5). The equation for the radius of curvature of an FRW universe containing radiation and a scalar Higgs field ϕ is[4]

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\left[\varepsilon_r\left(\frac{R_0}{R}\right)^4 + \varepsilon_\phi\right]\left(\frac{R}{c}\right)^2 = -c^2$$

where $\varepsilon_\phi = \frac{1}{2}(\partial\phi/\partial t)^2 + V_\phi(\phi)$. If compact dimensions exist, $\varepsilon_\phi \approx 0$ today, because the Friedmann equation (1) is a reasonable model for our universe.

Today, the "gravitational structure constant" $Gm_p^2/\hbar c = 5.91 \times 10^{-39}$ is the ratio of the strength of gravity to the strength of the strong force. The proton mass is $m_p = 1.67 \times 10^{-24}$ g, $\hbar = 1.05 \times 10^{-27}$ g cm^2/sec, the Planck mass $M = \sqrt{\hbar c/G} = 2.18 \times 10^{-5}$ g, and the Planck length $\delta = 1.62 \times 10^{-33}$ cm. Initially, gravity and the strong-electroweak (SEW) force had equal strength, $G_i m_p^2/\hbar c = GM^2/\hbar c = 1$, and the gravitational constant was $G_i = (M/m_p)^2\ G = 1.70 \times 10^{38}\ G$. The Planck length was $\delta_i = \sqrt{\hbar G_i/c^3} = (M/m_p)\ \delta = 2.11 \times 10^{-14}$ cm and the Planck mass was $M_i = \sqrt{\hbar c/G_t} = m_p$.

2. THE MODEL

This model is formulated in an abstract N-dimensional curvature space. This space describes the curvature of a homogeneous N-dimensional physical space with two subspaces related to the FRW and C systems. The coordinate in each of the N dimensions of a state in the curvature space is the radius of curvature of the corresponding dimension of that state in the N-dimensional physical space. Henceforth, the terms energy, angular momentum, force, and mass refer to effective quantities in this curvature space, unless indicated otherwise.

If the universe began spontaneously from a quantum fluctuation, all total quantum numbers must be zero. When the total energy and total angular momentum in the curvature space is zero, the Schrödinger equation for the N-dimensional radius of curvature is $-(\hbar^2/2m)\nabla_{\mathfrak{R}}^2\Psi + V_{\mathfrak{R}}\Psi = 0$ in the curvature space, where $\mathfrak{R}$ is the magnitude of an N-dimensional vector $\bar{\mathfrak{R}}$. Let $\mathfrak{R}^2 = R^2 + r^2$, where R is the radial coordinate in the 3-dimensional subspace describing the curvature of the isotropic FRW space and r is the radial coordinate in the $(n = N - 3)$-dimensional subspace describing the curvature of the C space. The separation of $V_{\mathfrak{R}}$ into terms involving only R and terms involving only r is suggested by the fact that the Friedmann equation with no explicit dependence on compact dimensions is a suitable model for our own (3 + 1)-dimensional universe. If $V_{\mathfrak{R}} = V_R + V_r$, the resulting Schrödinger equation

$$-\frac{\hbar^2}{2m}(\nabla_R^2 + \nabla_r^2)\Psi + (V_R + V_r)\Psi = 0$$

can be solved by separation of variables. The separated Schrödinger equation is

$$\left[\frac{1}{\psi(R)}\frac{-\hbar^2}{2m}\nabla_R^2\psi(R) + V_R\right] + \left[\frac{1}{\psi(r)}\frac{-\hbar^2}{2m}\nabla_r^2\psi(r) + V_r\right] = 0$$

where each square bracket equals a constant, denoted $-E$ and E, respectively.

This model assumes the Schrödinger equation for the radius of curvature R is the nonrelativistic quantum mechanical analog of the Friedmann equation for the radius of curvature of a closed radiation-dominated FRW universe. The Friedmann equation for a radiation-dominated universe (and the corresponding Schrödinger equation) is equivalent to the equation of motion of a nonrelativistic particle of mass m in the curvature space, moving with energy $-\frac{1}{2}mc^2$ in a $1/R^2$ potential. The equations can be generalized to energies E different from the Einstein value $E = -\frac{1}{2}mc^2$ (see, e.g., ref. 6, p. 744). The fact that the quantum dynamics of a radiation-dominated FRW universe is equivalent to the quantum dynamics of a particle in a $1/R^2$ potential is demonstrated in refs. 1 and 2 (although the limit on the strength of the potential in ref. 2 is inappropriate; see ref. 7, p. 1666). Note that this model aims only to explore some possible consequences of the just-mentioned equivalence, and the Schrödinger equation in the 3-dimensional subspace does *not* purport to represent the quantum mechanical extension of a classical field theory of gravity in a curved space-time.

The quantum mechanical analog of the generalized Friedmann equation is an S-wave Schrodinger equation in the 3-dimensional subspace of the N-dimensional curvature space. When $\varepsilon_\phi \approx 0$, the Schrödinger equation describing the curvature of the radiation-dominated physical FRW space (and reflecting the isotropy of the FRW universe) is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dR^2}\psi - \frac{4\pi mG}{3c^2}\frac{\varepsilon_r R_0^4}{R^2}\psi$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dR^2}\psi - \frac{mG}{2}\frac{A}{R^2}\psi = -\frac{mc^2}{2}\psi \tag{2}$$

Setting $\kappa = mc/\hbar$ and $p = \sqrt{m^2GA/\hbar^2 - 1/4}$ in equation (2), we find the solution to the Schrödinger equation for a radiation-dominated FRW universe to be $\psi = KRh_{ip-1/2}(i\kappa R)$, where h is the spherical Hankel function and K is a normalization constant (ref. 7, p. 1665). This radial wave function is zero at $R = 0$, so quantum mechanics forbids singularities in an FRW universe.

An S-wave Schrödinger equation must also be used to describe the curvature of the C space to make the total N-dimensional "angular momentum" zero, so the universe can arise spontaneously from a vacuum fluctuation. Thus, the C space must be isotropic in this model. Writing $\Psi = R^{-1}\psi(R)r^{-1/2(n-1)}\psi'(r)$, we find that the separated Schrödinger equation becomes

$$\left[\frac{1}{\psi(R)}\frac{-\hbar^2}{2m}\frac{d^2\psi(R)}{dR^2} + V_R\right] + \left[\frac{1}{\psi'(r)}\frac{-\hbar^2}{2m}\frac{d^2\psi'(r)}{dr^2} + \frac{\hbar^2(n-1)(n-3)}{8mr^2} + V_r\right] = 0 \tag{3}$$

The curvature of the universe is not changing rapidly today, so the forces now acting to change the curvature of the universe must be small. The radius of curvature of our three-dimensional universe is large today, so the $1/R^3$ force from radiation (and the $1/R^2$ force from matter) in the potential V_R is small. The radius of curvature of the C dimensions (if they exist) must be very small because they have not been observed. This indicates an inverse relationship between the forces acting to change the curvature of the two sectors of the universe, whereby the forces are small when r is small and R is large. So this model parametrizes the C-system potential as $V_r = kr^{\alpha}$, where α is a positive real number. It will be shown that $\alpha \approx 4$ is needed to produce our universe. Note that this model is not a generalized Kaluza–Klein model because it does not assume a generalized Einstein equation holds in the $(N + 1)$-dimensional universe, and the potential V_r is a monomial in r and not a polynomial in r^{-1}.[8]

The C-system effective potential

$$\frac{\hbar^2(n-1)(n-3)}{8mr^2} + kr^{\alpha} \tag{4}$$

has its minimum at the Planck length δ if $k = \hbar^2(n-1)(n-3)/(4m\alpha\delta^{2+\alpha})$. Then, at $r = \delta$,

$$V_r = \frac{\hbar^2(n-1)(n-3)}{4m\delta^2}\left(\frac{2+\alpha}{2\alpha}\right)$$

The ground state is at the bottom of the potential well, so

$$\frac{mc^2}{2} = \frac{\hbar^2(n-1)(n-3)}{4m\delta^2}\left(\frac{2+\alpha}{2\alpha}\right)$$

and $m = \beta\sqrt{\hbar c/G}$, where $\sqrt{\hbar c/G}$ is the Planck mass and

$$\beta = \tfrac{1}{2}\sqrt{(n-1)(n-3)(1 + 2/\alpha)}.$$

3. INITIAL CONDITIONS OF THE UNIVERSE

The universe can begin with a quantum fluctuation into a state composed of a radiation-dominated FRW universe obeying the Schrödinger equation

(2) with gravitational constant G_i, radius $\langle R\rangle = \delta_i$, and energy $-\frac{1}{2}\beta m_p c^2$, and the ground state of the C system with radius $\langle r\rangle = \delta_i$ and energy $\frac{1}{2}\beta m_p c^2$. If $V_\phi(\phi) = 0$ at the bottom of the C-system potential well, $\varepsilon_{\phi_i} = 0$ (because the radius of the C system does not change in this initial state, so $\partial\phi_i/\partial t = 0$). For small values of $p = \sqrt{m^2GA/\hbar^2 - 1/4}$, the probability density for the FRW radius is strongly peaked near δ_i and the initial value of the constant A in the Schrödinger equation (2) is just that required to produce $\langle R\rangle = \delta_i$. Then the initial state has $\langle R\rangle = \langle r\rangle = \delta_i$, the radius of curvature is the same in all N dimensions, and $dR/dt = dr/dt = 0$.

4. QUANTUM TUNNELING TRANSITION

Once the initial state arises by a quantum fluctuation from the vacuum, a quantum tunneling transition can occur from the initial state to another state with $\langle R\rangle = \langle r\rangle = \delta_i$, $dR/dt = dr/dt = 0$, and total energy zero, corresponding to the early stages of our universe, where the gravitational constant (and the Planck mass and Planck length) have today's value, the C-system effective potential (4) has its minimum at $r = \delta$, $k = \hbar^2(n-1)(n-3)/(4\beta M\alpha\delta^{2+\alpha})$, and ground state energy $\frac{1}{2}\beta Mc^2$, and the FRW system is described by the Schrödinger equation (2) with $m = \beta M$.

In particular, for one specific value of the constant A in equation (2), a tunneling transition can take the initial universe to a state where the C system is in an excited state of the effective potential (4) with energy E', $\langle r\rangle = \delta_i$, $dr/dt = 0$, and the FRW system is in a tightly bound non-Einstein state of the FRW Schrödinger equation (2) with energy $-E'$, $\langle R\rangle = \delta_i$, and $dR/dt = 0$. In this posttunneling state, the C system must be at its classical turning radius at $r = \delta_i$, with

$$E' = \frac{\hbar^2(n-1)(n-3)}{4\beta M\alpha\delta^2}\left(\frac{\delta_i}{\delta}\right)^\alpha$$

The FRW system must also be at its classical turning radius $R = \delta_i$, with A satisfying

$$-\frac{\beta MG}{2}\frac{A}{\delta_i^2} = -\frac{\hbar^2(n-1)(n-3)}{4\beta M\alpha\delta^2}\left(\frac{\delta_i}{\delta}\right)^\alpha$$

When the C system drops to its ground state with energy $\frac{1}{2}\beta Mc^2$ and raises the energy of the FRW system to the Einstein energy $-\frac{1}{2}\beta Mc^2$, the result is a radiation-dominated FRW universe with

$$A = \frac{\hbar}{c(1+0.5\alpha)}\left(\frac{\delta_i}{\delta}\right)^{\alpha+2}$$

The mechanism for the energy loss from the C system when the C system drops to its ground state and raises the FRW system to the Einstein energy must come from a more fundamental $(N + 1)$-dimensional model of the forces governing the universe. However, at the end of the process, the potential in the FRW Schrödinger equation (2) is

$$V_R = \frac{m}{2} \frac{1}{R^2} \left[\frac{G\hbar}{c(1 + 0.5\alpha)} \left(\frac{\delta_i}{\delta} \right)^{\alpha+2} \right]$$

If $\alpha = 4$, $V_R = 3.8\ (m/2)\ 10^{69}/\mathrm{R}^2$, near the approximate value $V_R = 5.6\ (m/2)\ 10^{71}/R^2$ estimated for our universe. Furthermore, since the fine structure constant is about 100 times smaller than the strong coupling constant, the breaking of the strong-electroweak symmetry might be expected to increase the value of A by about two orders of magnitude. So, this model suggests that a nonsingular radiation-dominated FRW universe similar to the early stages of our universe can be created by a quantum fluctuation from the vacuum (if our universe is closed).

5. INFLATION

The size of the C dimensions corresponds to a scalar field in the FRW system[5] that determines the strength of the gravitational constant and is constant throughout the FRW space. If the scalar field ϕ in the FRW system is related to r (the size of the C system)[5] by $\phi = \kappa(c/\sqrt{G} \ln(\delta_i/r)$, where κ is a real number, then $\phi = 0$ when $r = \delta_i$. The model has an inflationary phase because, after the quantum tunneling transition discussed above, the scalar field ϕ associated with the compact dimensions has a large energy density. The specifics of the inflation of the model universe after the tunneling transition depend on the rate of energy loss from the C system to the FRW system, dropping the C system to its ground state and raising the FRW system to the Einstein state; the rate of decay of the scalar field energy in the FRW system to radiation; and the value of the constant κ, which must be obtained from a more fundamental $(N + 1)$-dimensional model of the forces governing the universe. However, the inflationary process can be outlined by looking at the situation after the quantum tunneling transition in a little more detail.

Immediately after the quantum transition, $\phi = 0$, the gravitational constant is G_i and the Planck mass is m_p. Since $\varepsilon_\phi \approx 0$ today, $V_\phi\ (\phi_f) \approx 0$ when $r = \delta$ at the bottom of the potential well in the posttunneling state. Because the bottom of the effective potential (4) is no longer at $r = \delta_i$, $V_\phi(0) \neq 0$ after the tunneling transition. Since $V_\phi(0) \neq 0$ after the tunneling transition, $\varepsilon_\phi \neq 0$ and the FRW system can be described by the Schrödinger equation for a universe containing radiation and a scalar field:

$$-\frac{\hbar^2}{2m_\phi}\frac{d^2}{dR^2}\psi - \frac{4\pi m_\phi G_\phi}{3}(\varepsilon_r + \varepsilon_\phi)\left(\frac{R}{c}\right)^2\psi = -E'\psi$$

or

$$-\frac{\hbar^2}{2m_\phi}\frac{d^2}{dR^2}\psi - \frac{4\pi m_\phi G_\phi}{3c^2}\left(\frac{A'}{R^2} + \varepsilon_\phi R^2\right)\psi = -E'\psi$$

The effective potential in this equation has a local maximum at $R_{\max}^4 = A'/\varepsilon_\phi$. If the transition energy E' lies at the top of this local maximum, the quantum tunneling transition results in a non-Einstein FRW state (with wave packet centered at $R = \delta_i$) in unstable equilibrium (with $dR/dt = 0$). At the moment of transition $\varepsilon_\phi = A'/\delta_i^4$, so $\varepsilon_r = \varepsilon_\phi$ and A' must satisfy

$$\frac{8\pi m_\phi G_\phi A'}{3c^2\delta_i^2} = \frac{\hbar^2(n-1)(n-3)}{4\beta M\alpha\delta^2}\left(\frac{\delta_i}{\delta}\right)^\alpha$$

After the tunneling transition, when $R > R_{\max}$, the ε_ϕ term dominates and the universe inflates. In other words, after the tunneling transition, the force $(d/dr)V_r(\delta_i)$ constraining the C dimensions is greater than the zero force $(d/dR)V_R(\delta_i) = 0$ constraining the FRW dimensions. So, the C system shrinks, the energy of the C system drops to the ground-state energy, and the energy of the FRW system rises to the Einstein value $-\frac{1}{2}\beta Mc^2$. As the C system shrinks, ϕ increases, G decreases from G_i to its present value, the Planck mass increases from m_p to its present value (so $m_\phi \to \beta M$), and the energy in the scalar field is converted to radiation, doubling $\varepsilon_{\rm rad}$ as $\varepsilon_\phi \to 0$. The result is a radiation-dominated FRW universe satisfying equation (2) with

$$A = \frac{\hbar}{c(1 + 0.5\alpha)}\left(\frac{\delta_i}{\delta}\right)^{\alpha+2}$$

The classical analog of the FRW Schrödinger equation with an ε_ϕ term is

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3c^2}\left[\frac{A}{R^4} + \varepsilon_\phi\right] - \frac{E'}{R^2}$$

When R increases, the universe reaches a point where ε_ϕ dominates, and inflation begins. During inflation[4]

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \quad \text{and} \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$$

In this model the scalar field is necessarily the same at all points in 3-space and it changes simultaneously at all points in 3-space, avoiding problems with “bubble formation”,[8] inhomogeneity, or magnetic monopoles.[9] However, the SEW symmetries must break during inflation to prevent monopole dominance of the universe.

6. COSMOLOGICAL CONSTANT

After inflation, the scalar field ϕ corresponding to the compact dimensions is constant throughout 3-space. Limits on the cosmological constant[4] require the energy density of the scalar field to be $\varepsilon_\phi \approx 0$ (in spectacular disagreement with predictions of the standard model in quantum field theory[10]), but it is not exactly zero. After inflation, the energy density of the scalar field is determined by the zero-point energy of the Hamiltonian $\frac{1}{2}\dot{\phi}^2 + V_\phi(\phi)$. This nonzero energy density of the scalar field introduces a term analogous to a cosmological constant into the model, so the model universe eventually expands forever (ref. 4, p. 71). Cosmological observations establish an upper bound $\approx 10^{-54}$ cm^{-2} for the value of the cosmological “constant” today (ref. 4, p. 74). A lower bound on the cosmological “constant” today can be obtained from the uncertainty principle, because the energy ε_ϕ of the scalar field ϕ in a comoving unit volume cannot be precisely zero and $\Delta\varepsilon_\phi \Delta t \geq \hbar$. The time since inflation is $\approx 10^{17}$ sec (10^{10} years), so the uncertainty in the energy of the scalar field sets a lower bound on the effective cosmological “constant” of $\Lambda = 8\pi G \Delta\varepsilon_\phi / c^4 \geq 10^{-75}$ cm^{-2} today (ref. 4, p. 69).

REFERENCES

1. E. Elbaz, M. Novello, J. M. Salim, M. C. Motta da Silva, and R. Klippert, *Gen. Rel. Grav.* **29**, 481 (1997).
2. M. Novello, J. M. Salim, M. C. Motta da Silva, and R. Klippert, *Phys. Rev. D* **54**, 6202 (1996).
3. C. Magnan, *Comments Astrophys.* **17**, 285–295 (1994).
4. J. N. Islam, *An Introduction to Mathematical Cosmology*, Cambridge University Press, Cambridge (1992).
5. M. S. Turner, Cosmology and particle physics, in *Architecture of Fundamental Interactions at Short Distances*, P. Ramond and R. Stora, eds., Elsevier, Amsterdam (1987).
6. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* , Freeman, San Francisco (1972).
7. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York (1953).
8. Q. Shafi and C. Wetterich, *Phys. Lett.* **152**, 51–55 (1985).
9. S. Hawking and I. Moss, *Phys. Lett.* **110B**, 35–38 (1982).
10. S. Weinberg, *Rev. Mod. Phys.* **61**, 1–23 (1989).